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DYNAMICAL ANALYSIS OF A PREY-PREDATORMODEL WITH BEDDINGTON-DEANGELIS TYPE FUNCTION RESPONSE AND PREDATOR HARVESTING

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Abstract:
This paper discusses a prey-predator model with reserved area. The feeding rate of consumers (predators) per consumer is considered to be Beddington –DeAngelis type.The Beddington –DeAngelis Functional response but contains an extra term describing mutual interference by predators. In order to get more realistic models,refuge and linear harvesting have been incorporated. First,we find the existence of all possible equilibrium points and study the local stability properties of the proposed non-delayed model.In addition,for the non-delayed model,we perform the Hopf-bifurcation analysis around the interior equilibrium point based on the bifurcation parameters. Finally,numerical simulations are provided to verify the effectiveness of the proposed theoretical results.
Keywords: Predator harvesting, Equilibrium point, Beddington –DeAngelis Functional response, Stability, Hopf bifurcation.

1.**Introduction:**

Population biology is one of the interesting and applicable interdisplinary branches connecting Mathematics and biology.The origin and theory of this particular kind of dynamical system is dueTo pioneer work of Lotka. The intuitive and experimental observations also infer that the decrease in feeding rate of consumers(predators)per unit consumer is due to mutual inference among Predators[1-4]. The inclusion of interference among predators differentiates Beddington-DeAngelis Functional response from Holling type II Functional response. The Lotka-Volterra type predator- prey model with the Beddington-DeAngelis Functional response has been proposed and well studied.Motivated by these facts, we first propose A deterministic model with Beddington DeAngelis Functional response along with the harvesting factor of prey induced by predators and refuge. In the ecosystem,Predator and prey interaction is one of the most fundamental factors in shaping community structure and maintaining ecological diversity[5-

9].To capture the effects of predators on prey populations, two different approaches exist.

One is the consumption of prey (direct effect)by predators which is easier to observe in the field and has been the main focus of mathematical ecology.In general, mathematical models are classified into two main types namely ecology models and epi- demiological models. In ecological models studying the interactions between populations of a particular community are studied. Epidemiology models mean studying the spread of diseases between animalsand humans[10-14].It is increasingly crucial to do research on the dynamics of illness within ecological systems.The dynamical issues involved in the prey-predator mathematical model system can appear easy at first. Mathematical models are essential for understanding, studying, and investigating the expanse and management of infectious diseases . The effects of infectious diseases in a prey-predator system have been extensively studied[15-19].In the eco-epidemiological model, the infection of prey is highly significant. Lotka and Volterra's predator-prey models are considered important works in modern mathematical ecology in coupled systems of non-linear differential equations. Epidemiological models have attracted a lot of interest since Kermack-Mckendrick's pioneering work on SIRS because functional response is one of the most crucial elements in the prey-predator population.In environment nature,predators not only affect prey species through direct predation but also induce refuge.This refuge alters the behavior and reproductive patterns of prey individuals,impacting their survival and population dynamics.The refuge induced by by predators can lead to a reduction in the reproduction rate of prey.They may avoid open habitats the growth and sustainability of the prey population.In mathematical ecology,numerous species are living in ecosystems which engage in interactions that are influenced by various factors.These interactions can arise from defence mechanisms,response to natural disasters and human exploitation. As a result, maintaining a sustainable balance for the species within the ccosystem is cruial for long-term viability and ecological stability.The interplay between prey and predators are considered using the Beddington –DeAngelis Functional response.The feeding rate in the Holling type-II functional response decrease as the density of prey increases.However, the Beddington –DeAngelis Functional response considers both the effect of prey density on predators feeding rate and the impact of interference among predators depending on their density[20-23].By considering both of these factors,the Beddington –DeAngelis Functional response provides a more realistic model of the intricate relationships between predators and prey within the ecosystems.

In population dynamical and ecosystem,harvesting with refuge in a prey-predator system introduvces a behaviroral aspect that can influence the dynamics of an ecosystem.A predator-prey fishery model with stage structure and imprecise harvesting,considering interval uncertainty and the effects of refuge from the predator population on the prey.Also,they evaluated optimal harvesting strategy using Pontrygin’s maximal principle in an uncertain and imprecise setting.The proposed imprecise model formulation included variables related to refuge influence on prey population growth rate and the harvesting of prey and juvenile predator populations. Intially,they analyzed the model without delays to validate the positivity and boundesness of the solution[24-26].Further,they calculated the normal form of Hopf bifurcation to ascertain the direction and stability of bifurcated periodic solutions. Their finding were further supported through numerical simulations.

The paper is organized as follows: A mathematical model is developed in Section 2. Section 3 We discuss the positivity and boundedness of solutions. The stability analysis of the suggested model has been investigated in Section 4. For the proposed of Hopf Bifurcation in Section 5.In Section 6, numerical simulations of the suggested model are examined. Finally, the conclusion of the paper and the biological implications of our mathematical results are found in Section 7.

2. Mathematical Model Formation:

In this chapter ,we study the dynamics of a three species food web Eco-epidemiological model with Beddington DeAngelis Functional response with refuge and predator harvesting infected prey species of the form:

$$\left.\begin{aligned} \frac{dX}{dT} &= r_1X\left(1-\frac{X+Y}{K}\right)-\lambda YX-\frac{\alpha_1 XZ}{a_1+uX+vZ}, \\ \frac{dY}{dT} &= \lambda YX - d_1Y - \frac{b_1(1-m)YZ}{a_1+(1-m)Y}, \\ \frac{dZ}{dT} &= -d_2Z+\frac{Cb_1(1-m)YZ}{a_1+(1-m)Y}+\frac{C\alpha_1 XZ}{a_1+uX+vZ} \quad -HEZ \\ . \end{aligned}\right\} \qquad (1)$$

Subject to initial values $X(0) \geq 0, Y(0) \geq 0$ and $Z(0) \geq 0$.

Table 1: Biological representation of system (1) parameters

Parameters	**Biological Representation**	**Units**
X	Susceptible prey	Number per unit area (tons)
Y	Infected prey	Number per unit area (tons)
Z	Predator	Number per unit area (tons)
r_1	Intrinsic growth rate of prey	Per day (t^{-1})
K	Carrying capacity of environment	Number per unit area (tons)
α_1	Predation rate of Susceptible prey	Per day (t^{-1})
b_1	Predation rate of Infected prey	Per day (t^{-1})
a_1	Half saturation constant	m
C	Conversion rate of prey and predator	$0 \leq C \leq 1$
d_1	Death rate of prey	Per day (t^{-1})
d_2	Death rate of predator	Per day (t^{-1})
λ	Infection rate	Per day (t^{-1})
H	The catchability coefficient of the predator	Per day (t^{-1})
E	Harvesting effort	Per day (t^{-1})

To reduce system (1) parameters, adjust variables $x=\frac{X}{K}, y=\frac{Y}{K}, z=\frac{Z}{K}$and consider dimension time $t=\lambda KT$. Now, we apply the following transformations:

$$r=\frac{r_1}{\lambda k},\ \alpha=\frac{\alpha_1}{\lambda K},\ a=\frac{a_1}{\lambda K}, d=\frac{d_1}{\lambda K},\ \theta=\frac{b_1}{\lambda K}, \quad \delta=\frac{d_2}{\lambda k}, \mathrm{h}=\frac{HE}{\lambda k},$$

The equation (1) can be expressed in a non-dimensional form using the above transformations.

$$\frac{dx}{dt} = rx(1-x-y)-xy-\frac{\alpha xz}{a+ux+vz}$$
$$\frac{dy}{dt} = yx-dy-\frac{\theta(1-m)yz}{a+(1-m)y} \qquad (2)$$

$$\frac{dz}{dt} = -\delta z + \frac{c(1-m)\theta yz}{a+(1-m)y} + \frac{c\alpha xz}{a+ux+vz} - hz$$

Subject to initial values $x(0) \geq 0, y(0) \geq 0$ and $z(0) \geq 0$.

Solutions' positivity and Boundedness

This section explores the system's positivity and boundedness solution (2)

Positivity of solutions

THEOREM 3.1 ***All the solutions of (2) are positive in*** R^3_+

Proof. Since $x(0) \geq 0, y(0) \geq 0$, and $z(0) \geq 0$. hence the system (2.2) becomes,

$$x(t) = x(0)\exp\left(\int_0^1 [r(1-x-y) - xy - \frac{\alpha z}{a+ux+vz}]ds\right) \geq 0$$

$$y(t) = y(0)\exp\left(\int_0^1 [yx - dy - \frac{\theta(1-m)yz}{a+(1-m)y}]ds\right) \geq 0$$

$$z(t) = z(0)\exp\left(\int_0^1 [-\delta z + \frac{c(1-m)\theta yz}{a+(1-m)y} + \frac{c\alpha xz}{a+ux+vz} - hz]ds\right) \geq 0]$$

then the solutions of (2) are non-negative.

THEOREM 3.2 ***All the solutions of (2) are bounded in*** R^3_+

Proof: Since x(t),y(t),z(t) be any solution of the system (2) with positive initial conditions,since

$$\frac{dx}{dt} \leq rx(1-x)$$

We have,

lim sup $t \to \infty\, x(t) \leq 1$,

let w=x+y+z

$$\frac{dw}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}$$

$$= rx(1-x-y) - xy - \frac{\alpha\alpha z}{a+ux+vz} + yx - dy - \frac{\theta(1-m)yz}{a+(1-m)y} - \delta z + \frac{c(1-m)\theta yz}{a+(1-m)y}\frac{c\alpha xz}{a+ux+vz} - hz$$

$$= rx(1-x-y) - dx - \frac{(1-c)\alpha xz}{a+ux+vz} - hz - \frac{(1-c)\theta yz}{a+z} - \delta z$$

$$\leq rx(1-x-y) - dy - hz - \delta z \quad \text{since}(c<1)$$

$$\leq \frac{r}{4} - hx - dy - \delta z \quad \text{since } (\max(rx(1-x) = \frac{r}{4})$$

$$\leq \frac{r}{4} - \beta w \quad \text{where } \beta = \min(\delta, h)$$

$$\frac{dw}{dt} + \beta w \leq \frac{r}{4}$$

Using the differential inequality theory, then

$$0 < w \leq \frac{r}{4\beta}(1 - exp^{-\beta t}) + w(x_0, y_0, z_0,)\, exp^{-\beta t}$$

For $n \to \infty$,we have $0 < w \leq \frac{r}{4\beta}$.

Hence all the solutions of (2) is bounded for all the region, for $\varepsilon > 0$, then,

$\Omega = \ (x, y, z) \in R^3 ; x + y + z \leq \frac{r}{4\beta} + \epsilon)$

Equilibrium points and Stability analysis:

This section discusses the following possible equilibrium points of system (2).

- The trivial equilibrium point is $E_0(0,0,0)$.
- The infected free and predator free equilibrium point is $E_1(1,0,0)$..
- The predator free equilibrium point is $E_3(x, y, 0)$,

Where $x = d$ and $y = \frac{r(1-d)}{r+1}$.

- The interior equilibrium point is $E^*(x^*, y^*, z^*)$

where $y^* = \frac{a(a(\delta+h)(x-d)+(1-m)((\delta+h)-c\alpha)x^*}{(1-m)((c\alpha x^*+(c\theta-(\delta+h))(a+x^*))}$

$z^* = \frac{ac(x^*-d)(a+x^*)+(1-m)(r-rx^*)c}{(1-m)((c\alpha x^*+(c\theta-(\delta+h))(a+x^*))}$

and x^* is the only positive root of the equation for a quadratic Equation

$$Ax^2 + Bx + C = 0,$$

where $A = r\big(c\alpha + c\theta - (\delta + h)\big)$,

$$B = \big(c\theta - (\delta + h)\big)(-r + ar) - r\alpha c + a\big((\delta + h) + (d - c\alpha)r\big),$$

$$C = -a\big((r)\big(c\theta - (\delta + h)\big) + \big(c\alpha d - ad(1 + r)\big)\big).$$

4.1 Stability Analysis:

For the purpose of local stability analysis around various equilibrium points, we now want to compute the Jacobian matrix. The Jacobian matrix at an arbitrary point (u, v, w) is given by

$$J(x, y, z) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$a_{11} = r(1 - 2x) - i(r + 1) - \frac{\alpha az}{(a+ux+vz)^2},\ a_{12} = -x(r + 1),\ a_{13} = -\frac{\alpha x}{a+ux+vz}$$

$$a_{21} = y, a_{22} = x - d - \frac{a\theta z}{(a + y)^2}, a_{23} = \frac{-\theta y}{a + y},$$

$$a_{31} = \frac{\alpha caz}{(a+ux+vz)^2},\quad a_{32} = \frac{ac\theta z}{(a+y)^2},\quad a_{33} = \delta + \frac{c\theta y}{a+y} + \frac{c\alpha x}{a+ux+vz} - h$$

Theorem: 1

$E_0(0,0,0)$ is the trivial equilibrium point, which is saddle.

Proof:

The Jacobian matrix at E_0 is given by

$$J\,(E_0) = \begin{pmatrix} r & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -\delta - h \end{pmatrix}$$

The eigenvalues are $\lambda_1 = r, \lambda_2 = -d_1$ and $\lambda_3 = -\delta - h$

Hence, the equilibrium point E_0 is saddle.

Theorem : 2

$E_1(1,0,0)$ is the infected free and predator free equilibrium point which is unstable.

Proof:

The Jacobian matrix at E_1 is given by

$$J\,(E_1) = \begin{pmatrix} -r & -(r+1) & \frac{-\alpha}{a+1} \\ 0 & 1-d & 0 \\ 0 & 0 & -\delta + \frac{c\alpha}{a+1} - h \end{pmatrix}$$

The eigenvalues are $\lambda_1 = -r, \lambda_2 = 1 - d$ and $\lambda_3 = -\delta + \frac{c\alpha}{a+1} - h$

Due to numerical simulation table values, $1 - d$ is positive.

Hence, the equilibrium point E_1 is unstable.

Theorem: 3

$E_2(x, y, 0)$ is the predator free equilibrium point which is locally asymptotically stable if $d > c(\alpha + \theta)$.

Proof:

The Jacobian matrix at E_2 is given by

$$J\,(E_2) = \begin{pmatrix} b_{11} & b_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$b_{11} = d, b_{12} = (-1 - r)x, b_{13} = -\frac{\alpha}{a + ux + vy},$$
$$b_{21} = x, b_{22} = 0, b_{23} = \frac{\theta x}{a + ux + vy},$$
$$b_{31} = 0, b_{32} = 0, b_{33} = \frac{c\alpha x}{a + ux + vy} - d + \frac{c\theta y}{a + vy}.$$

The characteristic equation of the above Jacobian matrix is provided by

$$\lambda^3 + X\lambda^2 + Y\lambda + Z = 0.$$

Where,

$$X = -b_{11} - b_{22},$$
$$Y = -b_{31}b_{13} + b_{22}b_{11},$$
$$R = b_{13}b_{31}b_{12}.$$

According to Routh-Hurwitz criteria,

Hence, E_2 is locally asymptotically stable.

Theorem: 4

The interior equilibrium point $E^*(x^*, y^*, z^*)$ is locally asymptotically stable.

Proof:

The Jacobian matrix at E^* is given by

$$J\,(E^*) = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

Where ,

$$g_{11} = \frac{-x^*(-r + ar + (1 + r)y^* + 2rx^*)}{a + ux^* + vy^*}, g_{12} = -x^*(r + 1), g_{13} = -\frac{\alpha x^*}{a + ux^* + vy^*},$$
$$g_{21} = y^*, g_{22} = \frac{a\theta z^* y^*}{(a + v^*)^2}, g_{23} = \frac{\theta y^*}{a + y^*},$$
$$g_{31} = \frac{ac\alpha z^*}{(a + ux^* + vy^*)^2}, g_{32} = \frac{ac\theta z^*}{(a + y^*)^2}, g_{33} = 0.$$

The characteristic equation of the above Jacobian matrix is provided by

$$\lambda^3 + E\lambda^2 + F\lambda + G = 0. \qquad (4)$$

Where,

$$E = -g_{11} - g_{22},$$
$$F = -g_{31}g_{13} + g_{22}g_{11},$$

$G = g_{13}g_{31}g_{12}.$

According to Routh-Hurwitz criteria,

Hence, E^* is locally asymptotically stable.

5. Hopf Bifurcation analysis:

If $a_i(\alpha)$ i=1,2,3 are smooth functions of m in an open interval $m\epsilon\boldsymbol{R}$ such that the characteristic equation(2) has a pair of complex eigenvalues .The non dimensional system (3) experiences a Hopf bifurcation at the endemic equilibrium point E^* when bifurcation parameter α passes through the critical value $\alpha^* \in (0,1)$, provided that the following conditions are satisfied:

- the corresponding characteristic equation (3) of system (2) has a pair of complex conjugates $\lambda_{1,2} = a + ib,$ where $a > 0$ and one negative real root λ_3.
- $\left(\frac{dm(\alpha)}{d\alpha}\right)_{\alpha=\alpha^*} \neq 0.$

Here, we give the conditions under which a Hopf bifurcation would exist as the derivative's order approaches a critical value at the interior equilibrium point E^*.

6.Numerical Simulations:

In this part,we show some few numerical simulations on the system(2) to performed and validate the theoretical conslusions.The rate of refuge (m),and the rate of predation on susceptible prey α are major attributes used as control parameters in this study.for a given set of parameters values,numerical simulation is performed using the MATLAB software packages.Here a=0.2,r=0.5,d=0.1,θ=0.4,$\delta =$ 0.1,c=0.5, α =Variable, m=Variable.

Effect of varying the predation rate α

Figure (1) &(2) demonstrates an increase in the predation rate α decrease the population of infected prey and increase the predator population.

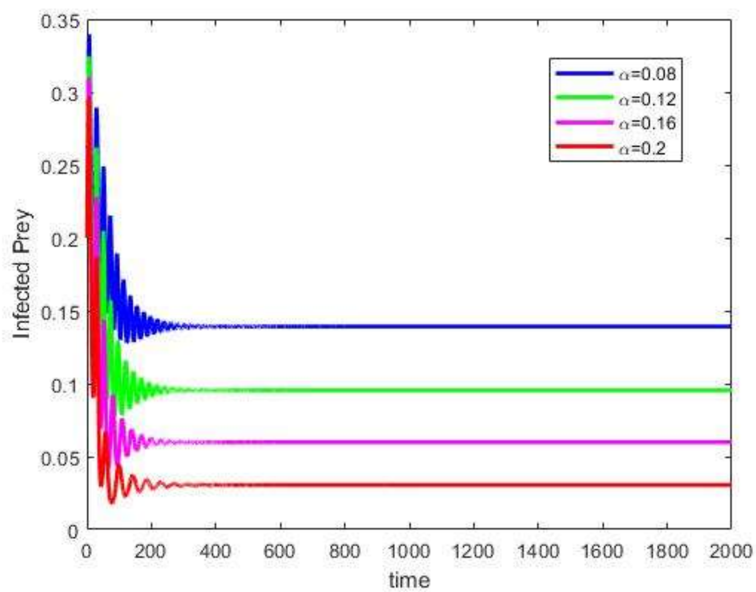

Figure 1: The population concentrations of infected prey population at system (2) equilibrium point E_2 and α=0.08, 0.12, 0.16, 0.2.

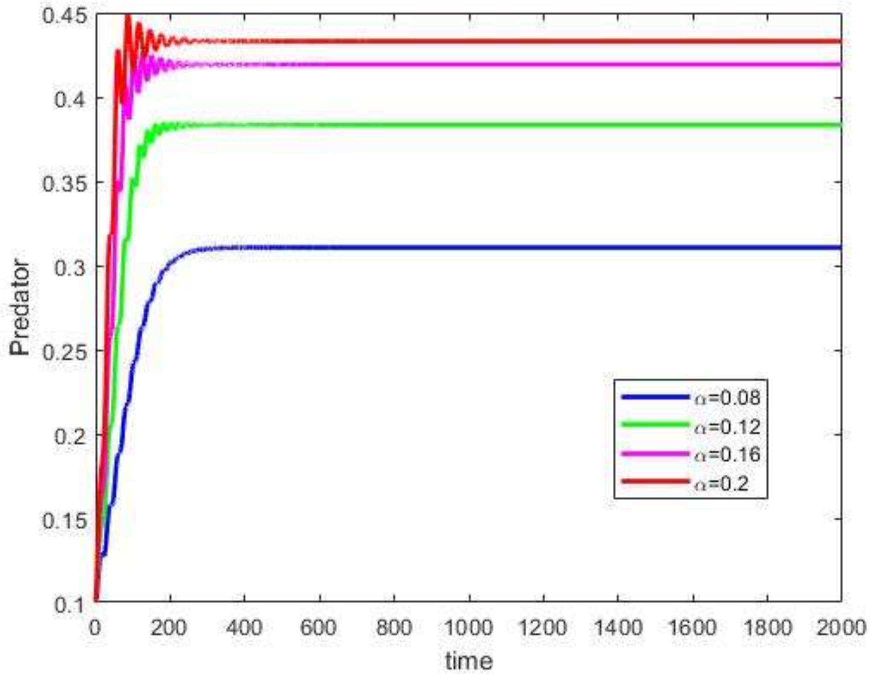

Figure 2: The population concentrations of predators population at system (2) equilibrium point E_2 and for α=0.08, 0.12, 0.16, 0.2

Effect of varying the refuge constant m

Figure (3) & (4) the density of the population of susceptible prey decreases as the refuge constant increases,we also an increase in the population of infected prey as the refuge constant increases from 0.1 to 0.3

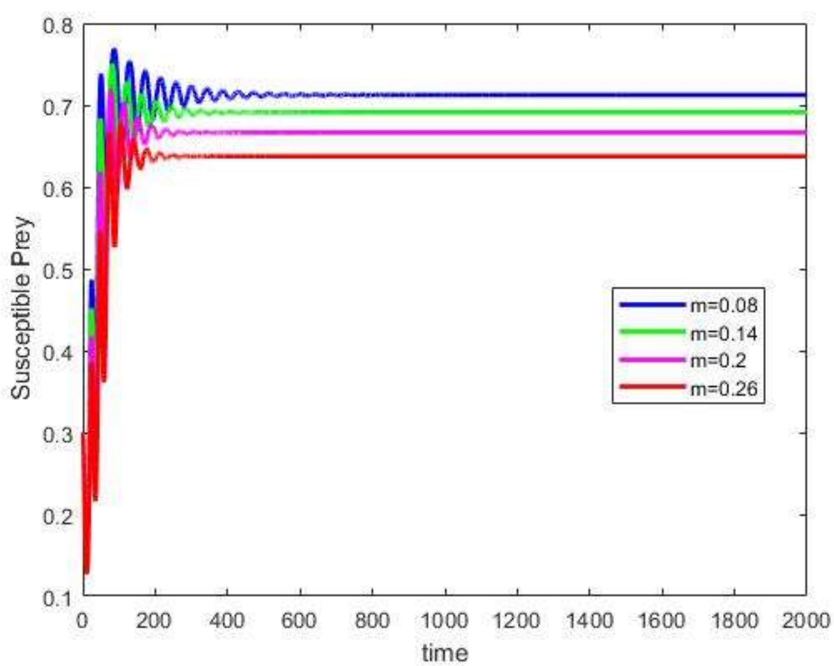

Figure 3: The population concentrations of susceptible prey population at system (2) equilibrium point E_2. and m=0.08,0.14,0.2,0.26

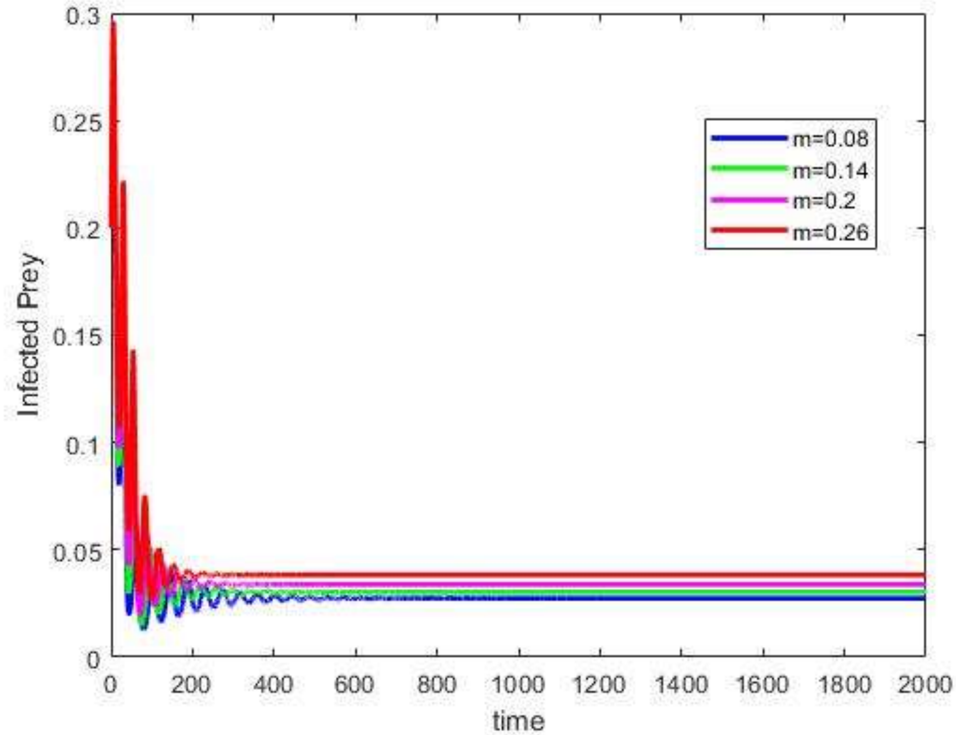

Figure 4: The population concentrations of infected prey population at system (2) equilibrium point E_2 and m=0.08,0.14,0.2,0.26

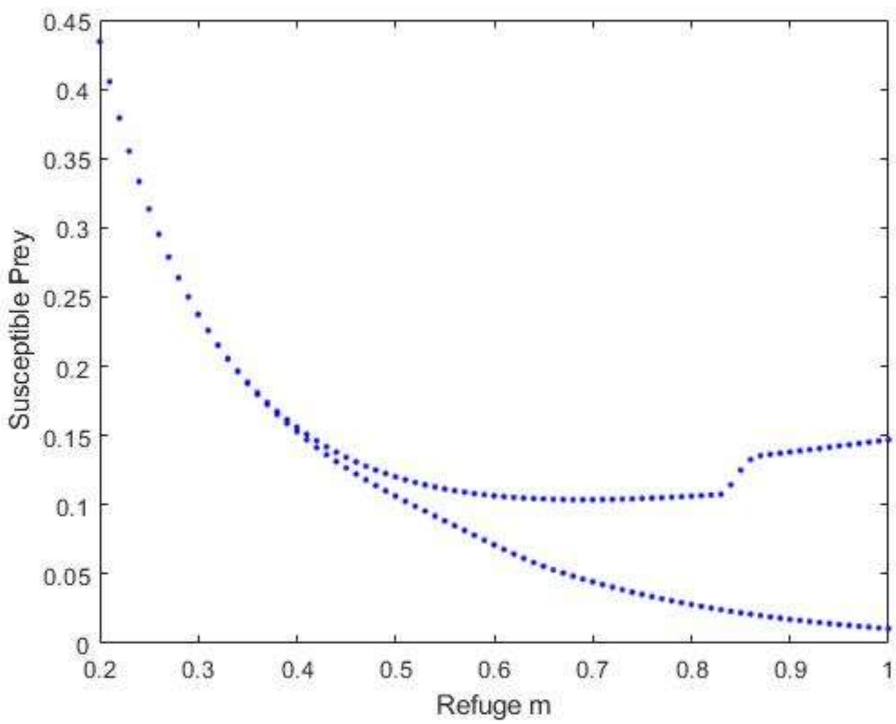

Figure 5: Bifurcation diagram at equilibrium point E^* of system (2) on the susceptible prey population.

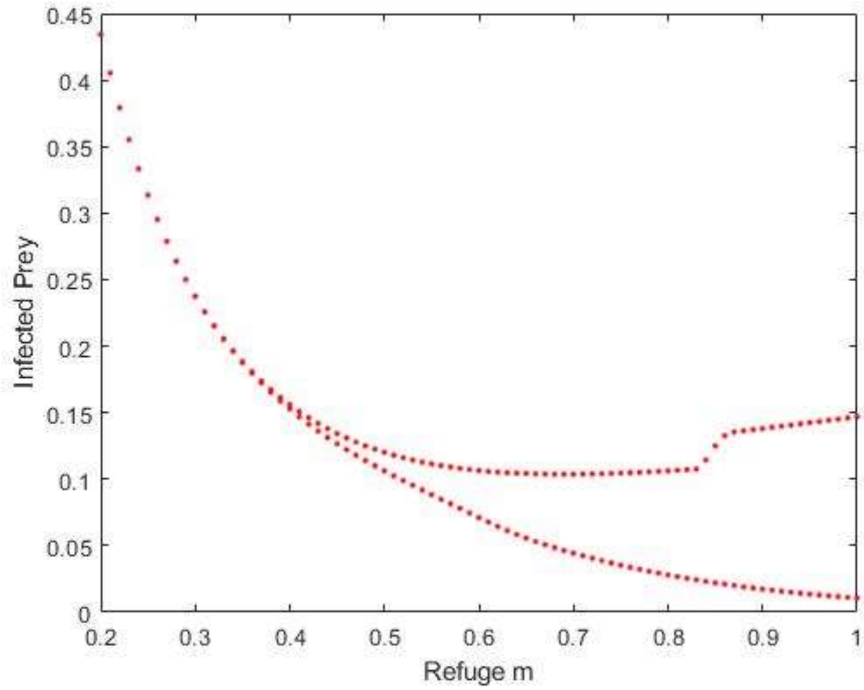

Figure 6: Bifurcation diagram at equilibrium point E^* of system (2) on the infected prey population.

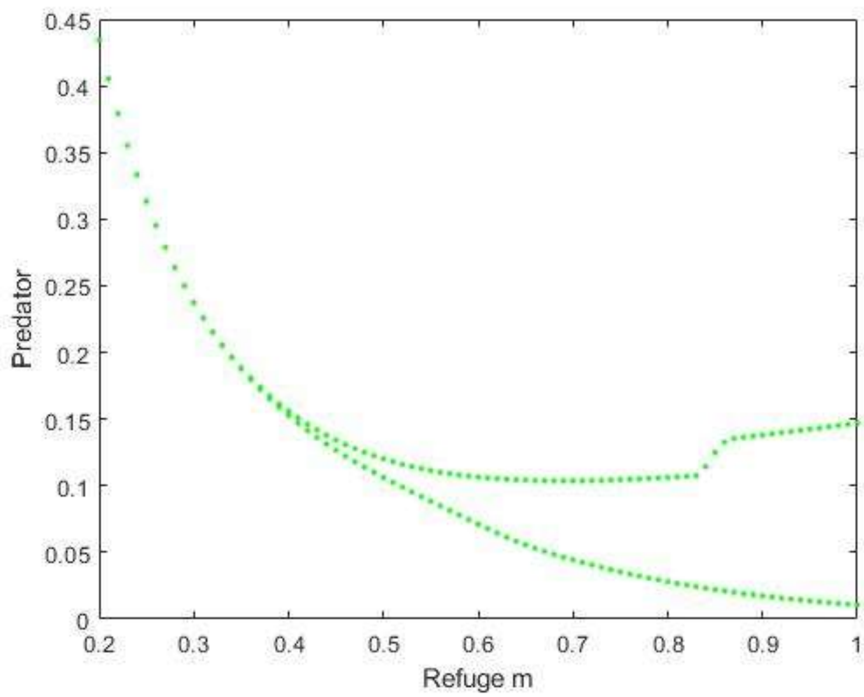

Figure 7: Bifurcation diagram at equilibrium point E^* of system (2) on the predator population.

Conclusion:

In the paper,we have developed a prey-predator model where only the prey population is being subjected to harvesting and the predator species is subjected to intra specific competition while both are under the effect of Beddington-DeAngelis functional response.Then we have discussed the dynamical behaviors of the system at various equilibrium points and their stability which are very similar to those of some recent research works.

The major difference between ourwork and the other recent work done is the incorporation of Beddington-DeAngelis functional response on a harvested prey species and a predator species under the effort of intra specific completion thereby enriching the dynamics of the system.we have further investigated the condition for limitcycle to arise by Hopfbifurcation.

In practice,multi-species system often exhibits more complex dynamical behaviors.For such system,we believe that there may be some similar results,which is interesting and left to our future work.

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